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A tri-Hamiltonian formulation of the self-induced transparency equations

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We present a tri-Hamiltonian formulation of the self-induced transparency (SIT) equations.

1. Introduction

Under certain conditions [1–3] an optical pulse can propagate through a nonlinear resonant medium in such a way that the leading edge of its intensity profile excites the medium, but its trailing edge de-excites the medium, precisely so as to leave behind no energy and, thus, to propagate without loss. This is self-induced transparency (SIT). Energy leaves the pulse, coherently excites the atoms, and then returns to the trailing edge of the pulse with no loss, but with a delay caused by the temporary storage of pulse energy in the atoms. (This delay shows up as an anomalously low pulse velocity, $\sim 10^{-3}c$.)

To the extent that resonant interaction of coherent light with a medium calls into play only a single atomic transition and the laser may be taken to be monochromatic, the medium has effectively only two levels. For sufficiently short pulse duration, the coherent interaction between the pulse and the medium leading to SIT may be taken to be lossless. (If the pulse is shorter than about 10 ns, it has no effective loss mechanisms, because it does not interact with any given group of atoms long enough for damping to take effect.) Certainly for most lasers and most atoms this two-level, lossless model can be an excellent approximation; and is quite adequate for an understanding of the basic physics behind many coherent transient phenomena. Indeed, the experimental work of Gibbs and Slusher [3] leaves little doubt about the validity of the SIT equations based upon this approximation [1,2], which we write in the form

$$(\partial_t - \partial)E = 2\langle P\rangle\ , \quad \partial_t P + 2\mathrm{i}\alpha P = 2DE\ ,$$
$$\partial_t D = -2\,\mathrm{Re}(EP^*)\ . \tag{1.1}$$

In these equations E and P denote the complex slowly-varying amplitudes of the self-consistent electric field and the polarizability of the medium, while D is the difference of its occupation numbers. The quantities E and P are complex scalar functions of space and time and D is a real function. The symbols ∂ and ∂_t denote partial derivatives with respect to space and time. The brackets $\langle\ \rangle$ denote averaging over a probability distribution $g(\omega)$ representing inhomogeneous frequency broadening, i.e., $\langle P\rangle = \int_{-\infty}^{\infty} P(x,t,\omega)g(\omega)\,\mathrm{d}\omega$. Finally, α represents the detuning between the transition frequency of the medium and the resonant cavity frequency.

Remarkably, the McCall–Hahn SIT equations fit into the integrable AKNS hierarchy [4,5], so the initial value problem for SIT is completely integrable. (See, e.g., ref. [6] for discussions of the integrable AKNS hierarchy and the SIT equations.) Thus, the SIT equations possess soliton solutions and are solvable on the real line in $1+1$ dimensions via the inverse scattering transform (IST). The earlier IST results will form the basis for our derivation of the tri-Hamiltonian formulation of the SIT equations. For convenience in what follows we treat the case in which detuning and inhomogeneous broadening are neglected, and write the SIT equations as

$$(\partial_t-\partial)E=2P\,,\quad \partial_t P=2DE\,,$$
$$\partial_t D=-2\,\mathrm{Re}(EP^*)\,. \tag{1.2}$$

Neglecting detuning and inhomogeneous broadening does not affect the existence of the tri-Hamiltonian formulation for SIT we derive here.

2. The spectral problem and Hamiltonian structures

We wish to construct the isospectral flows of the linear spectral problem

$$\lambda\psi_x=U\psi\equiv(\lambda^2A+\lambda U_1+U_0)\psi\,, \tag{2.1a}$$

where

$$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$U_i=\begin{pmatrix}w_i&q_i\\r_i&-w_i\end{pmatrix},\quad i=0,1\,. \tag{2.1b}$$

The SIT equations are known to be one of these isospectral flows, when $E=q_1$, $E^*=-r_1$, $P=q_0$, $P^*=r_0$, $D=-w_0$, and $w_1=0$. To demonstrate this, we seek the solution of the equation

$$\lambda V_x=[U,V]\,, \tag{2.2a}$$

in the form $V=\sum_{k=0}^{\infty}V_k\lambda^{-k}$. Substitution of this form into (2.2a) gives rise to a recursion relation for V_k:

$$V_{k+1,x}-[U_1,V_{k+1}]-[U_0,V_k]=[A,V_{k+2}]\,,$$
$$k\geqslant-2\,, \tag{2.2b}$$

whose off-diagonal and diagonal points must be treated separately. The off-diagonal part of V_{k+2} is immediately obtained by inverting $\mathrm{ad}\,A$. It would appear from (2.2b) that an integration is required to obtain the diagonal part of V_{k+1} so that our hierarchy would turn out to be nonlocal. However, (2.2a) implies that

$$(\mathrm{tr}\,V^2)_x=0\,, \tag{2.3a}$$

so that, by setting

$$\mathrm{tr}\,V^2=2\,, \tag{2.3b}$$

we can also determine the diagonal parts of V_k without any integration:

$$V_k^{\mathrm{diag}}=-\tfrac14\,\mathrm{tr}\sum_{i=1}^{k-1}V_iV_{k-i}\,. \tag{2.3c}$$

Thus, the given matrix V can be constructed in terms of *differential polynomials* of the coefficients (q_i, r_i, w_i). The first three coefficients are given by

$$V_0=A\,,\quad V_1=\begin{pmatrix}0&q_1\\r_1&0\end{pmatrix},$$
$$V_2=\begin{pmatrix}-\frac12 q_1r_1 & q_0-w_1q_1+\frac12 q_{1x}\\ r_0-w_1r_1-\frac12 r_{1x} & \frac12 q_1r_1\end{pmatrix}. \tag{2.4}$$

An infinite sequence of (polynomial) isospectral flows is obtained from the integrability conditions of (2.1) and

$$\psi_{t_m}=P_m\psi\,,\quad P_m=[\lambda^mV]_+=\sum_{i=0}^{m}V_{m-i}\lambda^i\,,$$
$$m=0,1,2,\ldots, \tag{2.5a}$$

which take the form

$$U_{t_m}=\lambda P_{mx}-[U,P_m]\,. \tag{2.5b}$$

The equations of motion for the matrices U_0 and U_1 are expressible as

$$U_{0t_m}=[U_0,V_m]\,,$$
$$U_{1t_m}=V_{mx}-[U_1,V_m]-[U_0,V_{m-1}]\,. \tag{2.6}$$

When $m=1$ (2.6) recovers the SIT equations (1.1) on the invariant subspace $w_1=0$.

Hamiltonian structures. Upon defining operators J_0, J_1 and J_2 by

$$J_0=-\mathrm{ad}\,U_0\,,\quad J_1=\partial-\mathrm{ad}\,U_1\,,$$
$$J_2=-\mathrm{ad}\,A\,, \tag{2.7}$$

the equations of motion (2.6) take the form

$$\begin{pmatrix} U_0 \\ U_1 \end{pmatrix}_{t_m} = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix} \begin{pmatrix} V_{m-1} \\ V_m \end{pmatrix} . \tag{2.8}$$

Using the recursion relation (2.2b), written in terms of the J's as

$$J_0 V_k + J_1 V_{k+1} + J_2 V_{k+2} = 0 , \tag{2.9}$$

allows the equations of motion to be written as

$$U_{t_m} = B_2 V^{(m)} = B_1 V^{(m+1)} = B_0 V^{(m+2)} , \tag{2.10a}$$

where

$$B_0 = -\begin{pmatrix} J_1 & J_2 \\ J_2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} J_0 & 0 \\ 0 & -J_2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}, \quad V^{(m)} = (V_{m-1}, V_m)^{\mathrm{T}} . \tag{2.10b}$$

In these formulae we use the trace to pair the vectors:

$$(\dot{q}_0, \dot{r}_0, \dot{w}_0, \dot{q}_1, \dot{r}_1, \dot{w}_1)^{\mathrm{T}}$$
$$\leftrightarrow (V^-_{m-1}, V^+_{m-1}, 2V^0_{m-1}, V^-_m, V^+_m, 2V^0_m) , \tag{2.11a}$$

where

$$V_k = \begin{pmatrix} V^0_k & V^+_k \\ V^-_k & -V^0_k \end{pmatrix} . \tag{2.11b}$$

The operators J_k now take the 3×3 form

$$J_0 = \begin{pmatrix} 0 & -2w_0 & q_0 \\ 2w_0 & 0 & -r_0 \\ -q_0 & r_0 & 0 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 0 & \partial - 2w_1 & q_1 \\ \partial + 2w_1 & 0 & -r_1 \\ -q_1 & r_1 & \frac{1}{2}\partial \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.12}$$

The operators B_k expressed this way are easily seen to be compatible Hamiltonian structures.

Remark. The forms of the operators B_k are identical to those presented in ref. [7]. However, in ref. [7] the operators J_k are defined by $J_k = \frac{1}{4}\epsilon_k \partial^3 + u_k \partial + \frac{1}{2} u_{kx}$.

In order to prove that the vectors $V^{(m)}$ are gradients we first need to check that this is true for $V^{(0)}$ and $V^{(1)}$. Indeed:

$$V^{(0)} = \delta\mathscr{H}_0 , \quad V^{(1)} = \delta\mathscr{H}_1 ,$$
$$\text{where } \mathscr{H}_0 = 2w_1 , \quad \mathscr{H}_1 = 2w_0 + q_1 r_1 . \tag{2.13a}$$

We can now invoke lemma 1 of the appendix to prove by induction the existence of an infinite sequence of functions $\mathscr{H}_m$, such that

$$V^{(m)} = \delta\mathscr{H}_m \quad \forall m \geqslant 0 . \tag{2.13b}$$

Taking these results together, we have

Theorem. The isospectral flows (2.5b) of the linear spectral problem (2.1a) are tri-Hamiltonian:

$$U_{t_m} = B_2 \delta\mathscr{H}_m = B_1 \delta\mathscr{H}_{m+1} = B_0 \delta\mathscr{H}_{m+2} . \tag{2.14}$$

Furthermore, the Hamiltonians $\mathscr{H}_m$ Poisson commute and hence, the isospectral flows of (2.1a) commute.

Remark. The Poisson commutivity of the $\mathscr{H}_m$ follows from lemma 2 of the appendix.

SIT reduction. It is easy to see that $\mathscr{H}_0$ is a common Casimir of both B_0 and B_1 (but not of B_2) so that B_0 and B_1 can *both* be restricted to the level surfaces of $\mathscr{H}_0$ by removing their bottom rows and right columns and setting w_1 constant. The SIT reduction corresponds to choosing $w_1 = 0$ (or to choosing $w_1 = \alpha$, when detuning is included) and is thus bi-Hamiltonian. The infinite sequence of Hamiltonians restrict to this submanifold. The first three of these are

$$\mathscr{H}_1 = 2w_0 + q_1 r_1 , \quad \mathscr{H}_2 = q_0 r_1 + q_1 r_0 - \tfrac{1}{2} q_1 r_{1x} ,$$
$$\mathscr{H}_3 = q_0 r_0 - w_0 q_1 r_1 - \tfrac{1}{4} q_1^2 r_1^2$$
$$+ \tfrac{1}{2}(r_0 q_{1x} - q_0 r_{1x} - \tfrac{1}{2} q_{1x} r_{1x}) . \tag{2.15}$$

The SIT equations are given by

$$U_{t_1} = B_1 \delta\mathscr{H}_2 = B_0 \delta\mathscr{H}_3 , \tag{2.16a}$$

which take the explicit form (cf. eq. (1.2)):

$$q_{0t_1} = -2w_0 q_1 , \quad r_{0t_1} = 2w_0 r_1 , \quad w_{0t_1} = q_1 r_0 - q_0 r_1 ,$$
$$q_{1t_1} = q_{1x} + 2q_0 , \quad r_{1t_1} = r_{1x} - 2r_0 . \tag{2.16b}$$

3. Traveling-wave solutions of SIT

Traveling-wave solutions of the SIT equations, defined by $U_i(x,t)=U_i(t-x)$, satisfy (after rescaling $U_0\to 2U_0$ and scaling the wave speed to unity) the system of ODEs

$$\dot q_0=-2w_0q_1\,,\quad \dot r_0=2w_0r_1\,,\quad \dot w_0=q_1r_0-q_0r_1\,,$$
$$\dot q_1=2q_0\,,\quad \dot r_1=-2r_0\,, \tag{3.1}$$

where an overdot denotes derivative with respect to $\tau=t-x$. The Lax representation for this system is easily derived from (2.5b), with $m=1$, by considering

$$\tilde L=U+\lambda P_1=2\lambda^2A+2\lambda U_1+U_0 \tag{3.2a}$$

and rescaling as above. After this rescaling, we are led to consider

$$L=\lambda^2A+\lambda U_1+U_0\,, \tag{3.2b}$$

where, for the moment, w_1 is not necessarily zero. In this case the centraliser of L (within sl(2)) is just L itself. Thus the isospectral flows of L are given by

$$L_{\tau_i}=[M_i,L],\quad i=0,1\,, \tag{3.3a}$$
$$M_0=(\lambda^{-2}L)_+=A\,,$$
$$M_1=(\lambda^{-1}L)_+=\lambda A+U_1\,. \tag{3.3b}$$

The traveling-wave SIT equations (3.1) correspond to M_1 (but with $w_1=0$). These isospectral flows are tri-Hamiltonian with operators D_i given by

$$D_0=-\begin{pmatrix}J_1 & J_2\\ J_2 & 0\end{pmatrix},\quad D_1=\begin{pmatrix}J_0 & 0\\ 0 & -J_2\end{pmatrix},$$
$$D_2=\begin{pmatrix}0 & J_0\\ J_0 & J_1\end{pmatrix}, \tag{3.4a}$$

where

$$J_0=-\operatorname{ad}U_0\,,\quad J_1=-\operatorname{ad}U_1\,,$$
$$J_2=-\operatorname{ad}A\,. \tag{3.4b}$$

The Hamiltonians corresponding to the isospectral flows are given by the coefficients of $\frac12\operatorname{tr}L^2$:

$$h_0=2w_1\,,\quad h_1=2w_0+q_1r_1+w_1^2\,,$$
$$h_2=q_0r_1+q_1r_0+2w_1w_0\,,\quad h_3=w_0^2+q_0r_0\,. \tag{3.5}$$

Remark. The operators D_i corresponds to Lie–Poisson brackets. When modified with the co-cycle:

$$\omega(x,y)=\lambda\int\operatorname{tr}(X\partial Y)\,\mathrm{d}x\,, \tag{3.6}$$

they give rise to the operators B_i of section 2.

Remark. We can obtain another sequence of Hamiltonians from those of (2.15). Since the latter give rise to local conservation laws for SIT:

$$\mathscr{H}_{nt_1}=\mathscr{F}_{n1x}\,, \tag{3.7a}$$

the traveling-wave property $U_1(x,t)=U_1(t-x)$ implies

$$\frac{\mathrm{d}}{\mathrm{d}\tau_1}(\mathscr{H}_n+\mathscr{F}_{n1})=0\,, \tag{3.7b}$$

where $\tau_1=t_1-x$. After the above rescaling, these constants are functionally related to those of (3.5).

As before the Hamiltonians (3.5) satisfy a (now finite) tri-Hamiltonian ladder relation:

$$D_0\nabla h_0=0\,,\quad D_1\nabla h_0=D_0\nabla h_1=0\,,$$
$$U_{\tau_0}=D_2\nabla h_0=D_1\nabla h_1=D_0\nabla h_2\,,$$
$$U_{\tau_1}=D_2\nabla h_1=D_1\nabla h_2=D_0\nabla h_3\,,$$
$$D_2\nabla h_2=D_1\nabla h_3=0\,,\quad D_2\nabla h_3=0\,. \tag{3.8}$$

Thus each operator has two Casimir functions. As before, the SIT reduction corresponds to setting $h_0=0$ and is bi-Hamiltonian, with operators D_0 and D_1.

Master symmetry. The translation $\lambda\to\lambda+s$ induces the transformation $L(\lambda)\to L(\lambda+s)$ given by

$$\bar U_0=U_0+sU_1+s^2A\,,\quad \bar U_1=U_1+2sA\,. \tag{3.9a}$$

This gives us a simple proof of compatibility of the Hamiltonian operators D_i, $i=0,1,2$. We define (where I is the 2×2 identity)

$$J(s)=\begin{pmatrix}I & -sI\\ 0 & I\end{pmatrix}, \tag{3.9b}$$

which is the Jacobian of the inverse of (3.9a), and use $D_i(s)$ to denote the operator D_i with U_k replaced by $\bar U_k$. We then have

$$J(s)D_2(s)J^{\mathrm T}(s)=D_2-2sD_1+s^2D_0\,,$$
$$J(s)D_1(s)J^{\mathrm T}(s)=D_1-sD_0\,,$$
$$J(s)D_0(s)J^{\mathrm T}(s)=D_0\,. \tag{3.10}$$

Since $D_2(\bar U_k)$ is Hamiltonian, the linear combination $D_2-2sD_1+s^2D_0$ is Hamiltonian for all s and thus D_0, D_1 and D_2 define *compatible* Poisson brackets. If we use $h_i(s)$ to denote h_i as a function of $\bar{U}_k$, then

$$h_3(s)=h_3+sh_2+s^2h_1+s^3h_0+s^4\,. \qquad (3.11)$$

Remark. Since h_3 is a Casimir of *both* D_1 and D_2, for *all* s, we can use (3.10) and (3.11) to give an alternative proof of relations (3.8). The infinitesimal generator of (3.9a) is the vector field

$$v=q_1\partial_{q_0}+r_1\partial_{r_0}+w_1\partial_{w_0}+2\partial_{w_1}\,. \qquad (3.12a)$$

Eqs. (3.10) and (3.11) then imply

$$L_vD_2=-2D_1\,,\quad L_vD_1=-D_0\,,\quad L_vD_0=0\,,$$
$$L_vh_3=h_2\,,\quad L_vh_2=2h_1\,,\quad L_vh_1=3h_0\,,$$
$$L_vh_0=4\,, \qquad (3.12b)$$

where L_v denotes the Lie derivative in the direction of v. A vector field which acts on Hamiltonians and Poisson brackets in this way is often called a master symmetry. note that, since the ∂_{w_1} component of v does not vanish on the $w_1=0$ surface, we cannot directly use this master symmetry in the SIT reduction.

Remark. A similar master symmetry exists in the original SIT partial differential equations (2.16b). However, we need to embed (2.1) into a more general class of spectral problem [8]:

$$(\epsilon_0+\epsilon_1\lambda)\psi_x=(\lambda^2A+\lambda U_1+U_0)\psi\,. \qquad (3.13)$$

Remark. The traveling-wave SIT equations are not invariant under the master-symmetry transformation. However, compatibility of its three Hamiltonian structures implies that the SIT equation may be expressed in several equivalent forms. Specifically, the ladder relations (3.8) imply the identity

$$U_{\tau_1}=(a_0D_2+a_1D_1+a_2D_0)$$
$$\times\nabla(A_0h_3+A_1h_2+A_2h_1+A_3h_0)\,,$$

provided the seven constants a_i, $i=0$, 1, 2, and A_j, $j=0$, 1, 2, 3, satisfy the two conditions

$$a_0A_2+a_1A_1+a_2A_0=1\,,\quad a_0A_3+a_1A_2+a_2A_1=0\,.$$

By linearly combining the constants of motion and Hamiltonian structures in this way, one may investigate various equivalent representations of SIT traveling-wave dynamics, which range from the complex Duffing equation to the spherical pendulum. See, e.g., ref. [9] for discussions of various aspects of the phase-space geometry of these dynamics and its chaotic response to periodic perturbations.

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Appendix

We present two basic lemmas which are referred to in the main body of this paper. A brief introduction to the theory of multi-Hamiltonian systems can be found in refs. [10,11].

A system of evolution equations is said to be bi-Hamiltonian if there exist two Hamiltonian operators B_0 and B_1 and two Hamiltonians $\mathscr{G}$ and $\mathscr{H}$ such that

$$u_t=B_0\,\delta\mathscr{G}=B_1\,\delta\mathscr{H}\,. \qquad (A.1)$$

It is particularly interesting if the operator B_0+B_1 is also Hamiltonian, in which case B_0 and B_1 are said to be compatible (in general the sum of the Poisson brackets would fail to satisfy the Jacobi identity). The importance of compatibility is that it enables us under certain conditions to construct an infinite hierarchy of (Poisson commuting) Hamiltonians. This important condition was first noticed by Magri [12], who proved the following pair of lemmas (also see ref. [10]):

Lemma 1. If B_0 and B_1 are compatible Hamiltonian operators, with B_0 nondegenerate, and

$$B_1\,\delta\mathscr{G}=B_0\,\delta\mathscr{H}\,,\quad B_1\,\delta\mathscr{H}=B_0K\,, \qquad (A.2)$$

then there exists a function $\mathscr{K}$ such that $K=\delta\mathscr{K}$.

To prove the existence of an infinite hierarchy of Hamiltonians, $\mathscr{H}_n$, related to compatible Hamiltonian operators B_0, B_1, we need to check that two conditions hold:

(i) $\exists$ an infinite sequence of vector functions K_0, K_1, ... satisfying

$$B_1 K_n = B_0 K_{n+1}\,, \tag{A.3}$$

(ii) $\exists$ two function(al)s $\mathscr{H}_0$ and $\mathscr{H}_1$ such that

$$K_0 = \delta\mathscr{H}_0\,,\quad K_1 = \delta\mathscr{H}_1\,.$$

It then follows from the lemma that there exist function(al)s $\mathscr{H}_n$ such that

$$K_n = \delta\mathscr{H}_n \quad \forall n \geqslant 0\,. \tag{A.4}$$

Lemma 2. Let $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$ denote the Poisson brackets defined respectively by B_0 and B_1, which are assumed to be compatible. Let $\mathscr{H}_0$, $\mathscr{H}_1$, ... be a sequence of functions defined by (A.3), A.4). Then these functions mutually commute:

$$\{\mathscr{H}_n, \mathscr{H}_m\}_0 = \{\mathscr{H}_n, \mathscr{H}_m\}_1 = 0 \quad \forall n, m \geqslant 0\,. \tag{A.5}$$

References

[1] S.L. McCall and E.L. Hahn, Phys. Rev. Lett. 18 (1967) 908.

[2] S.L. McCall and E.L. Hahn, Phys. Rev. 183 (1969) 457.

[3] H.M. Gibbs and R.E. Slusher, Phys. Rev. Lett. 24 (1970) 638; Phys. Rev. A 6 (1972) 2326.

[4] M.J. Ablowitz, D.J. Kaup and A.C. Newell, J. Math. Phys. 15 (1974) 1852.

[5] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. 53 (1974) 249.

[6] M.J. Ablowitz and H. Segur, Solitons and the inverse scattering transform (SIAM, Philadelphia, 1981) pp. 329–336.

[7] M. Antonowicz and A.P. Fordy, Physica D 28 (1987) 345.

[8] A.P. Fordy, Isospectral flows; their Hamiltonian structures, Miura maps and master symmetries, in: Applications of solitons, eds. P.J. Olver and D.H. Sattinger (IMA, 1990).

[9] D.D. Holm, G. Kovacic and B. Sundaram, Phys. Lett. A 154 (1991) 346.

[10] P.J. Olver, Applications of Lie groups to differential equations (Springer, Berlin, 1986).

[11] M. Antonowicz and A.P. Fordy, in: Soliton theory: a survey of results, ed. A.P. Fordy (Manchester Univ. Press, Manchester, 1990) pp. 273–312.

[12] F. Magri, J. Math. Phys. 19 (1978) 1156.